Zeros of exceptional Hermite polynomials

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Abstract

We study the zeros of exceptional Hermite polynomials associated with an even partition λ. We prove several conjectures regarding the asymptotic behaviour of both the regular (real) and the exceptional (complex) zeros. The real zeros are distributed as the zeros of usual Hermite polynomials and, after contracting by a factor $\sqrt{2n}$, we prove that they follow the semi-circle law. The non-real zeros tend to the zeros of the generalized Hermite polynomial H_λ, provided that these zeros are simple. It was conjectured by Veselov that the zeros of generalized Hermite polynomials are always simple, except possibly for the zero at the origin, but this conjecture remains open.

Keywords: Exceptional Orthogonal Polynomials, Hermite polynomials, Zero distribution

1. Introduction

The field of classical orthogonal polynomials is essentially the study of Sturm-Liouville problems with polynomial solutions. Indeed, by the well-known theorem of Bochner, if we assume that a Sturm-Liouville problem admits an eigenpolynomial of *every* degree, the we arrive at the well-known

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families of Hermite, Laguerre, Jacobi, and Bessel (if signed weights are allowed). Exceptional orthogonal polynomials arise when we consider Sturm-Liouville problems with polynomial eigenfunctions, but allow a finite number of degrees to be missing from the corresponding degree sequence. For background on exceptional orthogonal polynomials and exact solutions in quantum mechanics, see [10, 18]. For recent developments in the area of exceptional Hermite polynomials, see [6, 8].

The study of the zeros of exceptional orthogonal polynomials has attracted some recent interest. Preliminary results indicate that there are strong parallels with the behaviour of zeros of classical orthogonal polynomials. For example, it is possible to describe the zeros of certain exceptional OP using an electrostatic interpretation [5, 13]. Asymptotic behaviour of the zeros of 1-step Laguerre and Jacobi exceptional polynomials as the degree n goes to infinity was considered in [11, 14, 16].

All known families of exceptional OP have a weight of the form

$$W(x) = \frac{W_0(x)}{\eta(x)^2} \tag{1.1}$$

where $W_0(x)$ is a classical OP weight and where $\eta(x)$ is a certain polynomial whose degree is equal to the number of gaps in the XOP degree sequence, and which doesn't vanish on the domain of orthogonality. It has recently been shown [12] that the weight indeed takes the above form for every exceptional OP family.

The zeros of exceptional orthogonal polynomials are divided into two groups according to whether they lie in the domain of orthogonality. The *regular* zeros are of this type and enjoy the usual intertwining behaviour common to solutions of all Sturm-Liouville problems. All other types of zeros are called *exceptional zeros*. For sufficiently high degree n, the number of exceptional zeros is precisely equal to the degree of $\eta(x)$. Based on all extant investigations of the asymptotics of the zeros of exceptional OP it is reasonable to formulate the following.

Conjecture 1.1. *The regular zeros of exceptional OP have the same asymptotic behaviour as the zeros of their classical counterpart. The exceptional zeros converge to the zeros of the denominator polynomial $\eta(x)$.*

The first part of this conjecture admits two useful interpretations. The simplest interpretation is that after suitable normalization, the k-th regular zero of the exceptional polynomials converges to the k-th zero of their classical counterpart. Such asymptotic behaviour can be proved by means of Mehler-Heine type theorems, as was done for the case of certain exceptional Laguerre and Jacobi polynomials in [11] and [16].

However, there is another way to formulate this conjecture. It is well known that as the degree goes to infinity, the counting measure for the zeros of classical orthogonal polynomials, suitably normalized, tends to a certain equilibrium measure. The conjecture then is that the normalized counting measure of the regular zeros of exceptional orthogonal polynomials converges to the same equilibrium measure.

In this paper, we partially prove the conjecture for the class of exceptional Hermite polynomials [8]. Exceptional Hermite polynomials are Wronskians of $r+1$ classical Hermite polynomials, where r of the polynomials are fixed and where the degree of the last polynomial varies. The precise statement of the results is given in Section 2 after some necessary definitions. The proofs for the real zeros are given in Sections 3 and 4, and for the exceptional zeros in Section 5.

2. Exceptional Hermite polynomials

A partition λ of length r is a finite weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1),$$

The number $|\lambda| = \sum_{j=1}^{r} \lambda_j$ is the size of the partition. The generalized Hermite polynomial associated with λ is

$$H_\lambda = \mathrm{Wr}\left[H_{\lambda_r}, \ldots, H_{\lambda_2+r-2}, H_{\lambda_1+r-1}\right] \tag{2.1}$$

where Wr denotes the Wronskian determinant. The degree of H_λ is

$$\deg H_\lambda = |\lambda|.$$

Alternatively, we may also index the generalized Hermite polynomial by means of the strictly decreasing sequence $k_1 > k_2 > \cdots > k_r$ where $k_j = \lambda_j + r - j$, and then $H_\lambda = \mathrm{Wr}\left[H_{k_r}, \ldots, H_{k_2}, H_{k_1}\right]$.

We call λ an even (or double) partition if r is even and $\lambda_{2k-1} = \lambda_{2k}$ for $k = 1, 2, \ldots, r/2$. In that case it is known that H_λ has no zeros on the real line [1, 8, 15], and hence

$$W_\lambda(x) = \frac{e^{-x^2}}{(H_\lambda(x))^2}, \qquad x \in (-\infty, \infty), \tag{2.2}$$

is a well-defined weight function on the real line.

For a fixed partition λ, set

$$\mathbb{N}_\lambda = \{n \geq |\lambda| - r \mid n \neq |\lambda| + \lambda_j - j \text{ for } j = 1, \ldots, r\}, \tag{2.3}$$

and, for $n \geq |\lambda| - r$ set

$$P_n = \mathrm{Wr}\left[H_{\lambda_r}, H_{\lambda_{r-1}+1}, \ldots, H_{\lambda_2+r-2}, H_{\lambda_1+r-1}, H_{n-|\lambda|+r}\right]. \tag{2.4}$$

Then P_n is a polynomial of degree n if $n \in \mathbb{N}_\lambda$, and P_n vanishes identically otherwise. For this reason we call $\mathbb{N}_\lambda$ the degree sequence for λ. The complement $\mathbb{N}_0 \setminus \mathbb{N}_\lambda$ has the forbidden degrees. Their number are exactly $|\lambda|$. The largest forbidden degree is $|\lambda| + \lambda_1 - 1$.

It can be shown [8, Proposition 5.2] that the above polynomial P_n satisfies the second-order differential equation

$$P_n''(x) - 2\left(x + \frac{H_\lambda'}{H_\lambda}\right) P_n'(x) + \left(\frac{H_\lambda''}{H_\lambda} + 2x\frac{H_\lambda'}{H_\lambda} + 2n - 2|\lambda|\right) P_n(x) = 0. \tag{2.5}$$

Equivalently, we can say that $\varphi_n(x) = \frac{P_n(x)}{H_\lambda(x)} e^{-\frac{1}{2}x^2}$ is an eigenfunction of the differential operator

$$-y'' + \left(x^2 - 2\frac{d^2}{dx^2}\log H_\lambda\right) y \tag{2.6}$$

with eigenvalue $2n - 2|\lambda| + 1$. As a consequence, we can express the following weighted product as a perfect derivative:

$$P_n(x)P_m(x)W_\lambda(x) = \frac{d}{dx}\left[\frac{P_n(x)P_m'(x) - P_n'(x)P_m(x)}{2(n-m)} W_\lambda(x)\right], \quad n \neq m. \tag{2.7}$$

If λ is an even partition, then (2.7) implies the remarkable orthogonality property

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)W_\lambda(x)dx = 0, \qquad n, m \in \mathbb{N}_\lambda,\ n \neq m. \tag{2.8}$$

It is also known [11, 6] that the closure of the linear span of $\{P_n\}_{n\in\mathbb{N}_\lambda}$ is dense in the Hilbert space $\mathrm{L}^2(\mathbb{R}, W_\lambda)$. Because of this fact and because of (2.5) and (2.8), the polynomials P_n, $n \in \mathbb{N}_\lambda$ are called exceptional Hermite polynomials.

By the Sturm oscillation theorem, there are

$$n - |\lambda| + |\{j = 1, \dots, r \mid \lambda_j - j \geq n - |\lambda|\}|$$

real zeros of P_n; we call these the *regular* zeros. In particular, for $n \geq |\lambda| + \lambda_1$, there are exactly $n - |\lambda|$ regular (real) zeros. For $n \geq |\lambda| + \lambda_1$, let

$$x_{1,n} < x_{2,n} < \cdots < x_{n-|\lambda|,n} \tag{2.9}$$

denote the regular zeros of P_n arranged in ascending order. Since H_λ is an even polynomial, equation (2.5) has parity invariance, and consequently $P_n(x)$ has the same parity as n. The remaining $|\lambda|$ non-real zeros are called *exceptional* zeros; we will denote them as

$$z_{1,n}, z_{2,n}, \dots, z_{|\lambda|,n}. \tag{2.10}$$

We are now able to state our main results. Throughout the rest of the paper, λ is a fixed even partition. Our first result is the scaling limit of the central zeros of P_n.

Theorem 2.1. *For every fixed $k \in \mathbb{Z}$ we have*

$$\lim_{n\to\infty} 2\sqrt{n}\, x_{k+n+1-\frac{|\lambda|}{2},2n} = \frac{\pi}{2} + k\pi, \tag{2.11}$$

$$\lim_{n\to\infty} 2\sqrt{n}\, x_{k+n+1-\frac{|\lambda|}{2},2n+1} = k\pi. \tag{2.12}$$

Theorem 2.1 will follow from a Mehler-Heine asymptotic formula for P_n.

The next result is the weak scaling limit of the counting measure of the real zeros.

Theorem 2.2. *For every bounded continuous function f on $\mathbb{R}$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-|\lambda|} f\left(\frac{x_{j,n}}{\sqrt{2n}}\right) = \frac{2}{\pi} \int_{-1}^{1} f(x)\sqrt{1-x^2}dx. \tag{2.13}$$

Theorem 2.2 says that the normalized counting measure of the real zeros, when scaled by a factor $\sqrt{2n}$, tends weakly to the measure with density $\frac{2}{\pi}\sqrt{1-x^2}$ on $[-1, 1]$. This measure is known as the semicircle law. The corresponding result for the zeros of the Hermite polynomial H_n is well-known, see e.g. [4].

Our final result deals with the non-real zeros. We use $z_1, \dots, z_{|\lambda|}$ to denote the zeros of H_λ.

Theorem 2.3. *Let z_j be a simple zero of H_λ. Then there exists a constant $C > 0$ such that for all n large enough, there is a zero $z_{k,n}$ of P_n such that*

$$|z_j - z_{k,n}| \leq \frac{C}{\sqrt{n}}.$$

Theorem 2.3 proves the statement in Conjecture 1.1 on the exceptional zeros in the case that all zeros of H_λ are simple. In that case, we can relabel the exceptional zeros (2.10) in such a way that for every $j = 1, \ldots, |\lambda|$,

$$z_{j,n} = z_j + O\left(\frac{1}{\sqrt{n}}\right) \quad \text{as } n \to \infty,$$

and then clearly the sequence $(z_{j,n})_n$ tends to z_j for every j.

The condition of simple zeros may not be too restrictive, since it is actually believed that all non-real zeros of a generalized Hermite polynomial should be simple. In fact, Alexander Veselov made the following conjecture, that is quoted in [7]:

Conjecture 2.4. *For any partition λ the zeros of $H_\lambda(z)$ are simple, except possibly for the zero at $z = 0$.*

Our results are confirmed by numerical experiments. Figure 1 shows the twelve zeros of H_λ where $\lambda = (4, 4, 2, 2)$, which are simple and non-real. The figure also shows the zeros of the exceptional Hermite polynomial P_{40} of degree 40. It has 28 real zeros and 12 non-real zeros that are close to the zeros of H_λ as predicted by Theorem 2.3.

3. Proof of Theorem 2.1 and a Mehler-Heine formula

In this section we prove Theorem 2.1 by means of a Mehler-Heine formula for the polynomials P_n, which may be of interest in itself. This is a generalization of the classical Mehler-Heine formula for Hermite polynomials

$$\begin{aligned} \frac{(-1)^n \sqrt{n\pi}}{2^{2n} n!} H_{2n}\left(\frac{x}{2\sqrt{n}}\right) &\rightrightarrows \cos x, \\ \frac{(-1)^n \sqrt{\pi}}{2^{2n+1} n!} H_{2n+1}\left(\frac{x}{2\sqrt{n}}\right) &\rightrightarrows \sin x, \end{aligned} \tag{3.1}$$

where the double arrows denote uniform convergence on compact subsets of the complex plane as $n \to \infty$, see formulas 18.11.7 and 18.11.8 in [17].

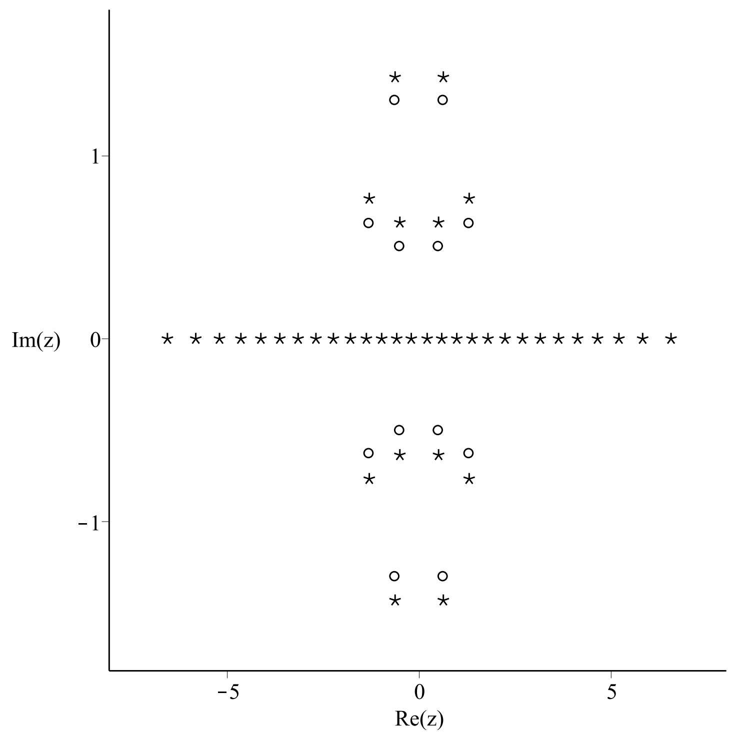

Figure 1: Plot of the zeros of H_λ with $\lambda = (4, 4, 2, 2)$ (open circles) together with the zeros of the corresponding exceptional Hermite polynomial of degree 40 (stars). Each zero of H_λ attracts exactly one non-real zero of P_n as $n \to \infty$.

Proposition 3.1. *Let λ be an even partition of length $r = 2s$ and $P_n(x)$, $n \in \mathbb{N}_\lambda$ the corresponding exceptional Hermite polynomials as defined in* (2.4)*. We then have*

$$\frac{(-1)^{n-\frac{|\lambda|}{2}}\sqrt{n\pi}}{2^{2n-|\lambda|+2r}(n-\frac{|\lambda|}{2}+\frac{r}{2})!}P_{2n}\left(\frac{x}{2\sqrt{n}}\right) \rightrightarrows H_\lambda(0)\cos x, \tag{3.2}$$

$$\frac{(-1)^{n-\frac{|\lambda|}{2}}\sqrt{\pi}}{2^{2n-|\lambda|+2r}(n-\frac{|\lambda|}{2}+\frac{r}{2})!}P_{2n+1}\left(\frac{x}{2\sqrt{n}}\right) \rightrightarrows H_\lambda(0)\sin x. \tag{3.3}$$

Proof. Expand the Wronskian determinant in (2.4) along the last column as

$$P_n = H_\lambda H^{(r)}_{n-|\lambda|+r} + \sum_{j=0}^{r-1} Q_j H^{(j)}_{n-|\lambda|+r} \tag{3.4}$$

where $Q_0 = \mathrm{Wr}[H'_{\lambda_r}, \dots, H'_{\lambda_1+r-1}]$ and in general, each coefficient Q_j is a differential polynomial in $H_{\lambda_r}, \dots, H_{\lambda_1+r-1}$ of degree

$$\deg Q_j = |\lambda| + j - r \tag{3.5}$$

that is independent of n.

Write

$$a_{2n} = \frac{(-1)^n\sqrt{n\pi}}{2^{2n}n!}, \qquad a_{2n+1} = \frac{(-1)^n\sqrt{\pi}}{2^{2n+1}n!}. \tag{3.6}$$

Then, since (3.1) can be differentiated any number of times by properties of uniform convergence of entire functions, we have for every non-negative integer j,

$$\frac{a_{2n}}{(2\sqrt{n})^j}H^{(j)}_{2n}\left(\frac{x}{2\sqrt{n}}\right) \rightrightarrows \left(\frac{d}{dx}\right)^j \cos x \tag{3.7}$$

$$\frac{a_{2n+1}}{(2\sqrt{n})^j}H^{(j)}_{2n+1}\left(\frac{x}{2\sqrt{n}}\right) \rightrightarrows \left(\frac{d}{dx}\right)^j \sin x \tag{3.8}$$

as $n \to \infty$, uniformly on compact subsets of the complex plane. Thus, since $|\lambda|$ and r are even,

$$\frac{a_{2n-|\lambda|+r}}{(2\sqrt{n})^r}H^{(r)}_{2n-|\lambda|+r}\left(\frac{x}{2\sqrt{n}}\right) \rightrightarrows (-1)^{r/2}\cos x$$

while, for $j = 0, \ldots, r-1$,

$$\frac{a_{2n-|\lambda|+r}}{(2\sqrt{n})^r} H^{(j)}_{2n-|\lambda|+r}\left(\frac{x}{2\sqrt{n}}\right) \rightrightarrows 0$$

From this we see that the first term in (3.4) is the dominant term as $n \to \infty$. Using the expression (3.6) for $a_{2n-|\lambda|+r}$ we get (3.2) after some simplifications.

Relation (3.3) is proved in an analogous fashion. □

Theorem 2.1 is an almost immediate consequence of Proposition 3.1.

Proof of Theorem 2.1. Recall that for $n \geq |\lambda| + \lambda_1$, the polynomial P_n has exactly $n - |\lambda|$ simple real zeros, and $|\lambda|$ non-real zeros.

It follows from (3.2) and Hurwitz's theorem from complex analysis, that those zeros of $P_{2n}\left(\frac{x}{2\sqrt{n}}\right)$ that do not tend to infinity, tend to the zeros of $\cos x$ as $n \to \infty$, with each zero of $\cos x$ attracting exactly one zero of $P_{2n}\left(\frac{x}{2\sqrt{n}}\right)$. Thus the non-real zeros of $P_{2n}\left(\frac{x}{2\sqrt{n}}\right)$ tend to infinity, and the real zeros tend to the zeros of $\cos x$, which gives (2.11).

The limit (2.12) follows from (3.3) in a similar way. □

4. Proof of Theorem 2.2

We start with a lemma.

Lemma 4.1. *Let $n \in \mathbb{N}_\lambda$. Then P_n is a linear combination of $H_n, \ldots, H_{n-s}$ where $s = |\lambda| + r$.*

Proof. From the expansion (3.4) and the fact that $H_n^{(j)}$ is a multiple of H_{n-j} we find

$$P_n(x) = \sum_{j=0}^{r} \tilde{Q}_j(x) H_{n-j}(x), \tag{4.1}$$

where $\tilde{Q}_j$ is a multiple of Q_j. Thus $\deg \tilde{Q}_j = |\lambda| + j - r$, see (3.5). If $k < n - |\lambda| - r$ then $\deg(x^k \tilde{Q}_j(x)) < n + j - 2r \leq n - j$ for $j = 0, 1, \ldots, r$. By the orthogonality of the Hermite polynomials we then have

$$\int_{-\infty}^{\infty} x^k \tilde{Q}_j(x) H_{n-j}(x) e^{-x^2} dx = 0, \qquad j = 0, 1, \ldots, r$$

and then also by linearity and (4.1)

$$\int_{-\infty}^{\infty} x^k P_n(x) e^{-x^2} dx = 0, \qquad k = 0, 1, \ldots, n - |\lambda| - r - 1.$$

This implies that P_n is a linear combination of $H_n, \ldots, H_{n-|\lambda|-r}$ as claimed in the lemma. $\square$

Let us recall the following known result [2, Theorem 3.1].

Theorem 4.2 (Beardon, Driver)**.** *Let $\{\pi_n\}_{n=0}^{\infty}$ be orthogonal polynomials associated with a positive measure. Fix $0 < s < n$ and let $c_{1,n} < \cdots < c_{n,n}$ be the zeros of π_n, listed in increasing order. Let P be a polynomial in the span of $\pi_s, \ldots, \pi_n$. Then, at least s of the intervals (c_k, c_{k+1}), $1 \leq k < n$ contain a zero of P.*

Corollary 4.3. *Let λ be an even partition of length r, and set $s = |\lambda| + r$. For $n > s$, let $c_{1,n} < \cdots < c_{n,n}$ be the zeros of the classical Hermite polynomial H_n, listed in increasing order. Then, at least $n - s$ intervals (c_k, c_{k+1}) contain a zero of the exceptional Hermite polynomial P_n.*

Thus as $n \to \infty$, at least $n - s$ of the zeros of P_n follow the zeros of the Hermite polynomial.

We are now able to give the proof of Theorem 2.2.

Proof of Theorem 2.2. Let f be a bounded continuous function on $\mathbb{R}$.

The normalized counting measure of the zeros of Hermite polynomials, scaled by the factor $\sqrt{2n}$, converges weakly to $\frac{2}{\pi}\sqrt{1-x^2}dx$, see [4]. Thus

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{c_{k,n}}{\sqrt{2n}}\right) = \frac{2}{\pi} \int_{-1}^{1} f(x)\sqrt{1-x^2}dx.$$

Then also, if $\xi_{k,n} \in (c_{k,n}, c_{k+1,n})$ for every $k = 1, \ldots, n-1$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{\xi_{k,n}}{\sqrt{2n}}\right) = \frac{2}{\pi} \int_{-1}^{1} f(x)\sqrt{1-x^2}dx. \tag{4.2}$$

Let $n > s = |\lambda| + r$. By Corollary 4.3, we can take $\xi_{k,n} \in (c_{k,n}, c_{k+1,n})$ to be a zero of P_n for every but at most s values of k. If we drop these at most s indices k from the sum in (4.2), then it will not affect the limit, since f is bounded. Then we have a sum over at least $n - s$ real zeros of P_n. Extending the sum by including the remaining real zeros (if any) of P_n does not affect the limit either, since their number is bounded by $s - |\lambda|$. Thus we obtain the limit (2.13). $\square$

5. Proof of Theorem 2.3

The following residue property is essential for the proof that follows.

Proposition 5.1. *Let λ be a partition and let P_n be the polynomial as defined in* (2.4)*. Then, the meromorphic function*

$$x \mapsto \frac{P_n(x)^2}{H_\lambda(x)^2} e^{-x^2}, \qquad x \in \mathbb{C}, \tag{5.1}$$

has vanishing residues at each of its poles (which are the zeros of H_λ).

Note that for $n \neq m$ the residues of $x \mapsto P_n(x)P_m(x)\frac{e^{-x^2}}{H_\lambda(x)^2}$ are zero as well, since by (2.7) this function is the derivative of a meromorphic function.

Proof. We consider the partition $\mu = (\lambda_2 \geq \cdots \geq \lambda_r)$ that is obtained from λ by dropping the first component. We denote by $\{P_{m,\mu}\}_{m\in\mathbb{N}_\mu}$ the sequence of polynomials associated with μ.

From the definitions (2.1) and (2.4) it is clear that $H_\lambda = P_{|\lambda|,\mu}$, and thus, as already noted in the paragraph containing (2.6),

$$\psi(x) := \frac{H_\lambda(x)}{H_\mu(x)} e^{-\frac{1}{2}x^2}$$

is an eigenfunction of the differential operator

$$L_\mu = -\frac{d^2}{dx^2} + \left(x^2 - 2\log H_\mu(x)\right),$$

with eigenvalue $2\lambda_1 + 1$.

Then by the properties of Darboux transformation, we have the factorization

$$L_\mu = A^\dagger A + (2\lambda_1 + 1) \tag{5.2}$$

with

$$A = -\frac{d}{dx} + \frac{\psi'(x)}{\psi(x)}, \qquad \text{and} \qquad A^\dagger = \frac{d}{dx} + \frac{\psi'(x)}{\psi(x)}, \tag{5.3}$$

and

$$L_\lambda := -\frac{d^2}{dy^2} + \left(x^2 - 2\log H_\lambda(x)\right) = AA^\dagger + (2\lambda_1 - 1). \tag{5.4}$$

In addition, if $\varphi \neq \psi$ is any other eigenfunction of L_μ, then $A\varphi$ is an eigenfunction of L_λ, and every eigenfunction of L_λ is obtained this way. Thus, associated with the polynomial $P_n = P_{n,\lambda}$ there is an index m such that

$$\varphi_{n,\lambda} = A\varphi_{m,\mu}, \tag{5.5}$$

where

$$\varphi_{n,\lambda}(x) = \frac{P_{n,\lambda}(x)}{H_\lambda(x)} e^{-\frac{1}{2}x^2}, \qquad \varphi_{m,\mu}(x) = \frac{P_{m,\mu}(x)}{H_\mu(x)} e^{-\frac{1}{2}x^2}.$$

Next, we get from (5.3) that

$$f(Ag) - (A^\dagger f)g = -\frac{d}{dx}(fg). \tag{5.6}$$

Taking $f = \varphi_{n,\lambda}$, $g = \varphi_{m,\mu}$ in (5.6), and noting (5.5) and

$$A^\dagger f = A^\dagger A g = (L_\mu - (2\lambda_1 + 1))g = cg$$

for some constant c, since g is an eigenfunction of L_μ, we obtain

$$\varphi_{n,\lambda}^2 - c\varphi_{m,\mu}^2 = -\frac{d}{dx}\left(\varphi_{n,\lambda}\varphi_{m,\mu}\right). \tag{5.7}$$

Now the proposition follows by induction on r. It is true if $r = 0$, since then $H_\lambda \equiv 1$, and (5.1) has no poles at all. Assuming the proposition is true for partitions of length $r-1$. Then the term $c\varphi_{m,\mu}^2$ in (5.7) has zero residues at each of its poles. Also the right-hand side of (5.7) has zero residues since it is the derivative of a meromorphic function. Thus $\varphi_{n,\lambda}^2$ has zero residues at each of its poles, which proves the proposition. $\square$

We are now ready for the proof of Theorem 2.3.

Proof of Theorem 2.3. Let $z_1, \ldots, z_{|\lambda|}$ be the zeros of H_λ and assume that z_j is a simple zero of H_λ. Without loss of generality we may assume that $\operatorname{Im} z_j > 0$. By Proposition 5.1,

$$\frac{P_n(x)^2}{\prod_{k\neq j}(x - z_k)^2} e^{-x^2},$$

then has a zero derivative at $x = z_j$. The logarithmic derivative is zero as well, which implies

$$2\frac{P_n'(z_j)}{P_n(z_j)} - 2\sum_{k\neq j}\frac{1}{z_j - z_k} - 2z_j = 0. \tag{5.8}$$

As before, let $x_{1,n}, \ldots, x_{n-|\lambda|,n}$ denote the real zeros of P_n, and $z_{1,n}, \ldots, z_{|\lambda|,n}$ the exceptional zeros. Then (5.8) tells us that

$$\sum_{k=1}^{n-|\lambda|}\frac{1}{z_j - x_{k,n}} + \sum_{k=1}^{|\lambda|}\frac{1}{z_j - z_{k,n}} = z_j + \sum_{k\neq j}\frac{1}{z_j - z_k} \tag{5.9}$$

From Plancherel-Rotach asymptotics of the Hermite polynomials, see e.g. [19, Theorem 8.22.9] one easily finds that for any compact interval $[a, b]$ on the real line, the number of zeros of H_n lying in $[a, b]$ grows roughly like $c\sqrt{n}$, as $n \to \infty$, for some constant $c > 0$. By Corollary 4.3, the same holds for the number of zeros of P_n in such an interval.

Any real zero $x_{k,n}$ in $[\operatorname{Re} z_j - 1, \operatorname{Re} z_j + 1]$ has $|z_j - x_{k,n}|^2 \leq 1 + (\operatorname{Im} z_j)^2$ and so

$$\frac{1}{|z_j - x_{k,n}|^2} \geq \frac{1}{1 + (\operatorname{Im} z_j)^2}. \tag{5.10}$$

Then for some constant $c_1 > 0$,

$$\operatorname{Im}\left(\sum_{k=1}^{n-|\lambda|}\frac{1}{z_j - x_{k,n}}\right) = -\sum_{k=1}^{n}\frac{\operatorname{Im} z_j}{|z_j - x_{k,n}|^2} < -c_1\sqrt{n}$$

since all terms in the sum have the same sign and at least $c\sqrt{n}$ of them satisfy (5.10). The right-hand side of (5.9) does not depend on n, and so it follows that for n large enough,

$$\sum_{k=1}^{|\lambda|}\operatorname{Im}\left(\frac{1}{z_j - z_{k,n}}\right) > c_1\sqrt{n}.$$

Since the number of terms does not depend on n, at least one of the terms is of order $\sqrt{n}$. Thus, for n sufficiently large, there is a non-real zero $z_{k,n}$ of P_n with

$$\operatorname{Im}\left(\frac{1}{z_j - z_{k,n}}\right) > c_2\sqrt{n}, \qquad c_2 = \frac{c_1}{|\lambda|}.$$

This is easily seen to imply that for this k,

$$|z_j - z_{k,n}| \leq \frac{1}{c_2\sqrt{n}}.$$

In fact, $\operatorname{Im} z_{k,n} > \operatorname{Im} z_j$ and $z_{k,n}$ lies in a circle centred at $z_j + \frac{i}{2c_2\sqrt{n}}$ with radius $\frac{1}{2c_2\sqrt{n}}$. Theorem 2.3 is proved. $\square$

6. Acknowledgements

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